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Relativistic theory of magnetoelastic interactions III. Isotropic media

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Abstract. In the third part of this work constitutive equations valid in *isotropic* media are obtained for both thermodynamically recoverable and dissipative processes. The reversible contributions are derived from the internal energy written in an *ad hoc* form corresponding to isotropy whereas the dissipative parts are derived from a convex dissipation potential. The former are nonlinear and describe elastic, magnetostrictive, magnetic anisotropy and exchange phenomena. The latter are quasi-linear and describe viscous stresses, heat and electricity conductions, and the damping of the magnetization precession. All classical known effects are included in the formulation at the nonrelativistic limit.

1. Introduction

The contents of this paper complement those of two foregoing papers (Maugin 1972a, 1973a, to be referred to as I and II respectively) in which the field equations and general constitutive equations for recoverable thermodynamical phenomena were given for a relativistically invariant theory of magnetoelastic interactions. Magnetic spins and gyromagnetic phenomena are taken into account in a continuous way. In this paper the emphasis is placed upon the study of *isotropic* media. This is the simplest but certainly not the most realistic degree of symmetry we can envisage. In fact, since the theory mainly concerns the behaviour of ferromagnetic materials, cubic structure (for example) would certainly be more adequate. However, the hypothesis of isotropy—which therefore implies the underlying assumption that the material is polycrystalline—allows us to carry out the study in a manageable (tensorial) mathematical form, for (i) we know representation theorems (Smith 1970, Spencer 1971) from which we can construct isotropic expressions that depend on a series of tensorial arguments—this is useful for representing the *nonlinear* constitutive equations corresponding to thermodynamically recoverable phenomena (elasticity with large deformation field, strong magnetization); (ii) it is a case for which *quasi-linear* constitutive equations which correspond to dissipative processes (viscous stresses, heat conduction, electric conduction, relaxation of the magnetization) can be given sufficiently simple forms, the theory being already quite complicated by itself.

A summary of the basic field equations obtained in I and basic notations are given in § 2. In § 3, constitutive equations that correspond to recoverable thermodynamical phenomena in *isotropic* solids are derived from the general constitutive equations obtained in II. It is shown that these equations can be constructed from eleven invariants if the functional form of the internal energy density is known. Although complicated, these equations offer a phenomenological representation, within the frame of an *exact* theory (ie, there are no approximations such as infinitesimal deformations, weak magnetization,...) of all expected effects such as elastic, magnetostrictive, magnetic anisotropy and exchange phenomena. The now classical Poynting and Kelvin effects of nonlinear elasticity (see Eringen 1962) are obviously included in the formulation, but they are not studied here. The linearization of the equations obtained is not performed, for it would be essentially similar to that made in the classical three-dimensional theory of micromagnetics (cf Maugin 1971c, 1972e, Maugin and Eringen 1972b). Section 4 deals with constitutive equations for *isotropic* dissipative materials. At this point, we follow the recent trend of continuum physics in considering a convex dissipation potential (essentially an extension of the ideas contained in the expression of the classical Rayleigh potential) from which constitutive equations for dissipative phenomena are derived. According to the recent formulation of the author (Maugin 1973, preprint), this is possible even for nonlinear constitutive equations. However, in the present case, it is difficult to postulate the form of an adequate dissipation potential. Therefore, in order to simplify the analysis, only the case of *quasi-linear* constitutive equations is examined. The dissipation potential considered assumes a 'natural' form justified in appendix 1. The sign of the corresponding material 'constants' is obtained by requiring this potential to be non-negative. The resulting dissipative phenomena are: (*a*) electrical conduction which corresponds to finite electrical conductivity; (*b*) a viscous (symmetric) stress which suggests that the theory could be applied to the study of liquid crystals in the nonrelativistic limit; (*c*) a dissipative skewsymmetric stress proportional to the difference between the angular velocity of the magnetization and the local rate of rotation of the deformable matter which, when interpreted, leads to a relaxation of the magnetization, that is, a damping of the magnetization precession, in a way similar to that described in the classical theory of micromagnetics (cf Maugin 1972d) and, therefore, includes in the limit the expressions given by Gilbert and Kelley (1955) and Landau and Lifshitz (1935); (*d*) dissipative couple stresses which yield a supplementary relaxation of the magnetization corresponding to hypothetical dissipative processes associated with exchange forces; heat conduction corresponding to a relativistic generalization of Fourier's law. However the latter yields a *parabolic* equation of heat flow which is unacceptable from the relativistic viewpoint. The answer to this paradox is provided in other works concerned with relativistic thermodynamics (Maugin 1973b, c). In cases (*c*) and (*d*) above, only the effect of the leading term is examined due to the complexity of the four-dimensional formulation.

From time to time, the comparison is made with the classical three-dimensional theory formerly studied by the author (Maugin 1971c, 1972d, 1973f, Maugin and Eringen 1972a, b). The equations established, especially those of § 3, will prove useful in a forthcoming study of wavefront propagation. In appendix 2, it is shown that several forms of the relativistic spin precession equation obtained in I–III are equivalent. The basic notation for relativistic continuum mechanics is to be found in Grot and Eringen (1966), Maugin (1971a, b), Maugin and Eringen (1972c).

2. Summary

The field equations which govern a magnetized deformable medium endowed with a continuous distribution of electronic spins and which are valid throughout a continuous region of the minkowskian space–time manifold M^4 have been given in the first part of the present work (Maugin 1972a). They are supplemented by thermodynamical equations and inequalities of which special forms were given in part II (Maugin 1973a). The relevant field equations are the continuity equation (I-4.1)†, the first and second Cauchy's equations (equations (I-4.2) and (I-4.5)), the energy equation (I-4.8) or (II-2.23) and the local entropy inequality (II-2.25) also referred to as the Clausius–Duhem inequality. These are supplemented by Maxwell's equations (I-4.10) through (I-4.14) which we shall not recall here. Thus, throughout a region $(\mathscr{B})$ of M^4 in which all fields are continuous, the set of field equations is the following:

$$(\rho u^\alpha)_{;\alpha} \equiv \dot{\rho} + \rho u^\alpha{}_{;\alpha} = 0, \tag{2.1}$$

$$\omega \dot{u}^\gamma = P^{\gamma\alpha} t^\beta{}_{\alpha;\beta} + \frac{1}{c^2} q \mathscr{E}^\gamma + \frac{1}{\mathrm{i}c^2} \epsilon^{\gamma\sigma\mu\nu} \mathscr{B}_\mu j_\sigma u_\nu + \rho \tilde{\mathscr{M}}_\lambda \mathscr{B}^{\lambda;\gamma} + \mathrm{O}(c^{-2}), \tag{2.2}$$

$$\mathbf{P}\left\{\rho \frac{\gamma^{-1}}{2} \dot{S}^{\alpha\beta}\right\} = t^{[\alpha\beta]} + \rho \tilde{\mathscr{M}}^{[\alpha} \mathscr{B}^{\beta]} + \mathbf{P}\{M^{\alpha\beta\gamma}{}_{;\gamma}\}, \tag{2.3}$$

$$\rho \dot{e} + \hat{q}^\beta{}_{;\beta} + \frac{1}{c^2} \hat{q}^\beta \dot{u}_\beta - t^{(\beta\alpha)} \sigma_{\alpha\beta} - t^{[\beta\alpha]} (\omega_{\alpha\beta} - \Omega_{\alpha\beta}) - M^{\alpha\beta\gamma} \mathscr{A}_{\beta\alpha\gamma} - \mathscr{E}_\gamma j^\gamma = -\rho h, \tag{2.4}$$

$$\frac{1}{\theta}\left(-\rho(\dot{\psi}^* + \eta \dot{\theta}) - \frac{1}{\theta} \hat{q}^\beta \mathring{\theta}_\beta + \mathscr{E}_\gamma j^\gamma + t^{(\beta\alpha)} \sigma_{\alpha\beta} + t^{[\beta\alpha]} (\omega_{\alpha\beta} - \Omega_{\alpha\beta}) + M^{\alpha\beta\gamma} \mathscr{A}_{\beta\alpha\gamma}\right) \geqslant 0. \tag{2.5}$$

In these equations we have defined

$$\omega \equiv \rho\left(1 + \frac{e}{c^2} - \frac{1}{c^2} \tilde{\mathscr{M}}_\alpha \mathscr{B}^\alpha\right), \tag{2.6}$$

$$S^{\alpha\beta} \equiv \frac{1}{\mathrm{i}c} \epsilon^{\alpha\beta\mu\gamma} \tilde{\mathscr{M}}_\mu u_\gamma, \qquad \tilde{\mathscr{M}}_\mu = \frac{1}{2\mathrm{i}c} \epsilon_{\mu\nu\alpha\beta} S^{\nu\alpha} u^\beta, \tag{2.7}$$

$$\mathring{\theta}_\beta \equiv P_\beta^{\cdot\gamma} (\theta_{,\gamma} + c^{-2} \theta \dot{u}_\gamma), \tag{2.8}$$

$$\sigma_{\alpha\beta} \equiv e_{(\alpha\beta)}, \qquad \omega_{\alpha\beta} \equiv e_{[\alpha\beta]}, \qquad e_{\alpha\beta} \equiv P_\alpha^{\cdot\mu} u_{\mu;\nu} P^\nu_{\cdot\beta}, \tag{2.9}$$

$$\mathscr{A}_{\beta\alpha\gamma} \equiv \mathbf{P}\left\{\Omega_{\beta\alpha;\gamma} + \frac{1}{c^2} \Omega_{\beta\alpha} \dot{u}_\gamma + \frac{2}{c^2} u_{[\beta;|\gamma|} \dot{u}_{\alpha]}\right\}. \tag{2.10}$$

The short hand notation $\mathbf{P}\{\ldots\}$ stands for the operation of projection on to a hypersurface locally orothogonal to the worldline of a particle, for example,

$$\mathbf{P}\{\dot{S}^{\alpha\beta}\} = P^\alpha_{\cdot\mu} P^\beta_{\cdot\nu} \dot{S}^{\mu\nu}, \qquad P^\alpha_{\cdot\mu} \equiv \delta^\alpha_\mu + \frac{1}{c^2} u^\alpha u_\mu, \tag{2.11}$$

$P^\alpha_{\cdot\mu}$ being the projection operator (cf I). The other symbols introduced in the equations given above bear the following significance‡: c: light velocity in vacuum; ρ: proper density of matter; u^α: four velocity; $\dot{u}^\alpha$: four acceleration; $t^{\beta\alpha}$: relativistic stress tensor (PU); $M^{\alpha\beta\gamma}$: relativistic couple stress tensor (PU); $\mathscr{E}^\alpha$: four electric field (PU); $\mathscr{B}^\alpha$: four magnetic induction (PU); j^α: four conduction current (PU); $\tilde{\mathscr{M}}^\alpha$: four magnetization per unit of proper mass (PU); $S^{\alpha\beta}$: magnetization bi-vector (dual of $\tilde{\mathscr{M}}^\alpha$) (PU); γ: gyromagnetic

† Equations of part I and part II are referred to with a prefix I and II respectively.

‡ The abbreviation PU introduced in part I (for *perpendicular to* u^α) indicates that the quantities so labelled are essentially spatial. Their projections along the worldline direction vanish (ie, their purely time-like and their mixed space–time components vanish). For example, $A_{\alpha\beta}$ is said to be completely PU if and only if $A_{\alpha\beta} u^\alpha = A_{\alpha\beta} u^\beta = 0$.

ratio; q : volumic electric charge; θ : proper thermodynamical temperature; $\overset{*}{\theta}_\alpha$: relativistic temperature gradient (PU); $\hat{q}^\alpha$: heat conduction four vector (PU); h : heat source per unit of proper mass; ω : total energy per unit of proper volume; e : specific internal *magneto*-energy (that is, e depends on $\tilde{\mathscr{M}}^\alpha$ and not on $\mathscr{B}^\alpha$); η : specific entropy; ψ^* : *magneto*-free energy per unit of proper mass; $\sigma_{\alpha\beta}$: relativistic rate of strain tensor (PU); $\omega_{\alpha\beta}$: relativistic rate of rotation or vorticity tensor (PU). $\mathscr{A}_{\beta\alpha\gamma}$ is the complicated kinematical PU quantity defined by equation (2.12). Finally $\Omega_{\alpha\beta}$ is the angular (PU) velocity of the magnetization $\tilde{\mathscr{M}}^\alpha$ so that the proper time evolution of $\tilde{\mathscr{M}}^\alpha$ is given by†

$$\dot{\tilde{\mathscr{M}}}_\alpha = \left(\Omega_{\alpha\beta} + \frac{1}{c^2} u_\alpha \dot{u}_\beta\right) \tilde{\mathscr{M}}^\beta. \tag{2.12}$$

The second term in the parentheses represents a Fermi–Walker transport of $\tilde{\mathscr{M}}^\alpha$ along the worldline of the 'particle' equipped with $\tilde{\mathscr{M}}^\alpha$. In the right hand side of equation (2.2), we recognize the classical contributions: (*a*) the divergence of the mechanical stress tensor, (*b*) the Lorentz force, (*c*) the Stern–Gerlach force. The term $O(c^{-2})$ stands for terms which vanish at the nonrelativistic limit $c \to \infty$ (cf full formula in I).

In the formulae above, semicolons denote covariant derivatives, a superposed dot indicates propertime differentiation, parentheses around a set of indices denote symmetrization and brackets denote alternation. Indices enclosed between vertical bars are not antisymmetrized. $\epsilon^{\alpha\beta\gamma\delta}$ is the permutation symbol of which the following algebra formulae will be useful:

$$\epsilon^{\mu\nu\sigma\tau}\epsilon_{\mu\nu\alpha\beta} = 2(\delta^\sigma_\alpha\delta^\tau_\beta - \delta^\sigma_\beta\delta^\tau_\alpha), \tag{2.13}$$

$$\epsilon^{\alpha\lambda\gamma\rho}\epsilon_{\alpha\beta\mu\nu} = \delta^\lambda_\beta\delta^\gamma_\mu\delta^\rho_\nu - \delta^\lambda_\beta\delta^\rho_\mu\delta^\gamma_\nu + \delta^\rho_\beta\delta^\lambda_\mu\delta^\gamma_\nu - \delta^\rho_\beta\delta^\gamma_\mu\delta^\lambda_\nu + \delta^\gamma_\beta\delta^\rho_\mu\delta^\lambda_\nu - \delta^\gamma_\beta\delta^\lambda_\mu\delta^\rho_\nu. \tag{2.14}$$

It is also useful to notice that $\Omega_{\alpha\beta}$, $M^{\alpha\beta\gamma}$ and $\mathscr{A}_{\beta\alpha\gamma}$ satisfy the symmetry conditions:

$$\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}, \qquad M^{\alpha\beta\gamma} = -M^{\beta\alpha\gamma}, \qquad \mathscr{A}_{\beta\alpha\gamma} = -\mathscr{A}_{\alpha\beta\gamma}. \tag{2.15}$$

The quantities appearing in equations (2.1) through (2.4) for which we need constitutive equations are $t^{\beta\alpha}$, $M^{\alpha\beta\gamma}$, q^β and j^γ. The first and the second of these in general present recoverable and dissipative parts. The two last ones resort to purely dissipative phenomena, heat and electrical conductions. Thus we have

$$t^{\beta\alpha} = {}^{\mathrm{R}}t^{\beta\alpha} + {}^{\mathrm{D}}t^{\beta\alpha}, \qquad M^{\alpha\beta\gamma} = {}^{\mathrm{R}}M^{\alpha\beta\gamma} + {}^{\mathrm{D}}M^{\alpha\beta\gamma}, \tag{2.16}$$

where the left superscripts R and D stand for *recoverable* and *dissipative* respectively. Constitutive equations for ${}^{\mathrm{R}}t^{\beta\alpha}$ and ${}^{\mathrm{R}}M^{\alpha\beta\gamma}$ are obtained for nonlinear *isotropic* elastic materials in the following section. Possible constitutive equations for ${}^{\mathrm{D}}t^{\beta\alpha}$ and ${}^{\mathrm{D}}M^{\alpha\beta\gamma}$ in *isotropic* materials are derived in § 4.

3. Constitutive equations for isotropic non-dissipative solids

We have established in part II (Maugin 1973a) that the recoverable parts ${}^{\mathrm{R}}t^{\beta\alpha}$ and ${}^{\mathrm{R}}M^{\alpha\beta\gamma}$ corresponding to the behaviour of a hyperelastic medium were derivable from the Lorentz invariant potential

$$\psi^* = \psi^*(C_{KL}, M_L, M_{LK}, \theta, X^K), \tag{3.1}$$

† To make easier the interpretation in the subsequent developments, we have made the change $\Omega_{\alpha\beta} \to -\Omega_{\alpha\beta}$ in equation (II-1.6), hence a similar alteration in the terms involving $\Omega_{\alpha\beta}$ in equations (2.4), (2.5) and (2.10).

with

$$\psi^* \stackrel{\text{def}}{=} e - \eta\theta, \qquad \eta = -\frac{\partial \psi^*}{\partial \theta} \tag{3.2}$$

and

$$C_{KL} \equiv x_{\alpha K} x^{\alpha}_{.L}, \qquad x^{\alpha}_{.K} u_{\alpha} = 0, \tag{3.3}$$

$$M_L \equiv x^{\alpha}_{.L} \tilde{\mathscr{M}}_{\alpha}, \qquad \tilde{\mathscr{M}}_{\alpha} u^{\alpha} = 0, \tag{3.4}$$

$$M_{LK} \equiv x^{\alpha}_{.L} M_{\alpha K} = x^{\alpha}_{.L} \tilde{\mathscr{M}}_{\alpha;\mu} x^{\mu}_{.K}, \qquad M^{\alpha}_{.K} u_{\alpha} = 0. \tag{3.5}$$

It is clear after equation (3.2) that, instead of ψ^*, one may use the potential†

$$e = \bar{e}(C_{KL}, M_L, M_{LK}, \eta, X^K) \tag{3.6}$$

whose functional dependence is obtained from (3.1) via the Legendre transformation (3.2, part I). Then

$$\theta = \frac{\partial \bar{e}}{\partial \eta}, \tag{3.7}$$

and the constitutive equations (II-3.41) and (II-3.42) take the forms

$$^{\mathrm{R}}t^{\beta\alpha} = \rho\left(2\frac{\partial \bar{e}}{\partial C_{KL}} x^{\alpha}_{.K} + \frac{\partial \bar{e}}{\partial M_L} \tilde{\mathscr{M}}^{\alpha} + \frac{\partial \bar{e}}{\partial M_{LK}} \tilde{\mathscr{M}}^{\alpha}_{.;\mu} x^{\mu}_{.K}\right) x^{\beta}_{.L}, \tag{3.8}$$

$$^{\mathrm{R}}M^{\gamma\beta\mu} = \rho \frac{\partial \bar{e}}{\partial M_{LK}} \tilde{\mathscr{M}}^{[\gamma} x^{\beta]}_{.L} x^{\mu}_{.K}. \tag{3.9}$$

However, in view of further studies (in a forthcoming part IV devoted to the study of wave propagation), it seems more convenient to use a different—but still equivalent‡—set of constitutive arguments in equation (3.6) and to write

$$e = \tilde{e}(\overset{-1}{C}{}^{KL}, \hat{M}^L, \hat{M}^{LK}, \eta, X^K) \tag{3.10}$$

in which $\overset{-1}{C}{}^{KL}$ is the reciprocal to C_{KL} (cf part I), that is,

$$\overset{-1}{C}{}^{KL} C_{LM} = \delta^K_M, \qquad K, L, \ldots = 1, 2, 3. \tag{3.11}$$

Further we have defined

$$\hat{M}^L \equiv M_K \overset{-1}{C}{}^{KL}, \qquad \hat{M}^{KL} \equiv M_{MN} \overset{-1}{C}{}^{KM} \overset{-1}{C}{}^{LN}. \tag{3.12}$$

It is readily shown that

$$\hat{M}^L = \tilde{\mathscr{M}}_{\alpha} X^{L,\alpha}, \qquad \hat{M}^{KL} = X^{K,\gamma} \tilde{\mathscr{M}}_{\gamma;\delta} X^{L,\delta}. \tag{3.13}$$

† The functions defined by equations (3.1) and (3.6) obey the principle of material indifference in relativity enunciated in Maugin (1972b, c).

‡ The new set of constitutive arguments forms a minimal function basis for $\tilde{e}$ as C_{KL}, M_L and M_{LK} formed one for $\bar{e}$. In other words, the set of arguments that appear in $\tilde{e}$ is but an equivalent solution to the partial differential equation (II-3.6).

Noting that, if $\mathscr{D}$ denotes any differential operator, it follows from equation (3.11) that

$$\mathscr{D}\overset{-1}{C}{}^{KJ} = -\overset{-1}{C}{}^{KL}\overset{-1}{C}{}^{MJ}\mathscr{D}C_{LM}$$

we can replace the set of equations (3.8)–(3.9) by

$$^{\mathrm{R}}t^{\beta\alpha} = \rho\left(-2\frac{\partial\tilde{e}}{\partial\overset{-1}{C}{}^{MN}}X^{M,\alpha}+\frac{\partial\tilde{e}}{\partial\hat{M}^{N}}\tilde{\mathscr{M}}^{\alpha}+\frac{\partial\tilde{e}}{\partial\hat{M}^{NM}}\tilde{\mathscr{M}}^{\alpha}_{\cdot;\mu}X^{M,\mu}\right)X^{N,\beta}, \tag{3.14}$$

$$^{\mathrm{R}}M^{\gamma\beta\mu} = \rho\frac{\partial\tilde{e}}{\partial\hat{M}^{MN}}\tilde{\mathscr{M}}^{[\gamma}X^{M,\beta]}X^{N,\mu}. \tag{3.15}$$

3.1. *Case of hemitropic homogeneous hyperelastic media*

This case is of particular interest since it corresponds to the simplest symmetry that one may consider. Furthermore it is the only one for which we know representation theorems (cf Smith 1970, Spencer 1971) that yield an explicit formulation of the nonlinear constitutive equations (3.14) and (3.15). To start with, we remark that the arguments of $\tilde{e}$ are scalars in M^4. They however are tensorial quantities in the reference state $(B_{\mathrm{R}}) \subset E^3_{\mathrm{R}}$ of the material body. The material symmetry is studied in this reference state, the three-dimensional euclidean space E^3_{R} (a space-like section of M^4 at a certain time) being equipped with the symmetric metric G_{AB} which is used to raise and lower capital Latin indices.

Homogeneity—which requires invariance under the group of translations $\{\boldsymbol{B}\}$ in E^3_{R}—implies that

$$\frac{\partial\tilde{e}}{\partial X^K} = 0. \tag{3.16}$$

Hemitropy implies invariance of $\tilde{e}$ under the proper orthogonal group $\{\boldsymbol{S}\}$ in E^3_{R}. A theorem due to Smith (1970), applied in Maugin and Eringen (1972b), then states that if N is the number of independent components of the tensorial arguments of $\tilde{e}$, the minimal function basis for $\tilde{e}$ is built up of $N-p$ members, with $p = n(n-1)/2$ where n is the dimensionality of space. For *hyperelastic* media, we need not worry about the uniqueness of those members (cf Maugin 1971c). In the present case†, on account of (3.16),

$$e = \tilde{e}(\overset{-1}{C}{}^{KL}, \hat{M}^{L}, \hat{M}^{LK}).$$

There are apparently eighteen independent tensorial components $(\overset{-1}{C}{}^{KL}:6, \hat{M}^{L}:3, \hat{M}^{LK}:9)$, but the saturation of the magnetization that we assume in the present study implies that

$$\tilde{\mathscr{M}}^{\alpha}\tilde{\mathscr{M}}_{\alpha} = \text{constant}, \qquad \tilde{\mathscr{M}}^{\alpha}\tilde{\mathscr{M}}_{\alpha;\mu} = 0 \tag{3.17}$$

which represent four constraints (the second of equation (3.17) is not independent of the first, for, multiplied by u^{μ}, it yields the proper time derivative of the first). In terms of the arguments of $\tilde{e}$, equations (3.17) can be restated as

$$\hat{M}_{L}\overset{-1}{C}{}^{LN}\hat{M}_{N} = \text{constant}, \qquad \hat{M}_{A}\overset{-1}{C}{}^{AB}\hat{M}_{BN} = 0. \tag{3.18}$$

† We omit the dependence upon η, for it is only used to define the thermodynamical temperature (cf equation (3.7)).

Those constraints amount to four. Hence we have $N = 18-4$ and

$$N\text{-}p = 14-\frac{3(3-1)}{2} = 11.$$

The eleven members of the minimal function basis of $\tilde{e}$, for hemitropic materials, can be chosen among the members of the integrity basis listed in Spencer (1971). $\hat{M}^L$ is axial, hence we introduce its dual Σ_{AB} in E_{R}^3

$$\Sigma_{AB} \equiv \epsilon_{ABC}\hat{M}^C, \qquad \hat{M}^L = \tfrac{1}{2}\epsilon^{LPQ}\Sigma_{PQ}. \tag{3.19}$$

Note also that, since $\boldsymbol{\Sigma}$ is skewsymmetric and $\overset{-1}{\boldsymbol{C}}$ is symmetric (Tr = trace)

$$\mathrm{Tr}\,\boldsymbol{\Sigma} = \mathrm{Tr}\,\overset{-1}{\boldsymbol{C}}\!:\!\boldsymbol{\Sigma} = \mathrm{Tr}\,(\overset{-1}{\boldsymbol{C}})^2\!:\!\boldsymbol{\Sigma} = 0. \tag{3.20}$$

Moreover the trace of $\boldsymbol{\Sigma}^3$ is not independent of $\mathrm{Tr}\,\boldsymbol{\Sigma}^2$ after the Cayley–Hamilton theorem. Then selecting the following list of invariants:

$$\left.\begin{aligned}
&I_{(1)} = \mathrm{Tr}\,\overset{-1}{\boldsymbol{C}}, \qquad I_{(2)} = \tfrac{1}{2}\mathrm{Tr}\,(\overset{-1}{\boldsymbol{C}})^2, \qquad I_{(3)} = \tfrac{1}{3}\mathrm{Tr}\,(\overset{-1}{\boldsymbol{C}})^3,\\
&I_{(4)} = \tfrac{1}{2}\mathrm{Tr}\,\boldsymbol{\Sigma}^2, \qquad I_{(5)} = \mathrm{Tr}\,\boldsymbol{\Sigma}^2\!:\!\overset{-1}{\boldsymbol{C}}, \qquad I_{(6)} = \tfrac{1}{2}\mathrm{Tr}\,\boldsymbol{\Sigma}^2\!:\!(\overset{-1}{\boldsymbol{C}})^2\\
&I_{(7)} = \mathrm{Tr}(\hat{M}^K_{\cdot L}), \qquad I_{(8)} = \tfrac{1}{2}\mathrm{Tr}(\hat{M}^K_{\cdot L}\hat{M}^L_{\cdot P}),\\
&I_{(9)} = \tfrac{1}{3}\mathrm{Tr}(\hat{M}^K_{\cdot L}\hat{M}^L_{\cdot N}\hat{M}^N_{\cdot P}), \qquad I_{(10)} = \mathrm{Tr}(\overset{-1}{C}{}^K_{\cdot L}\hat{M}^L_{\cdot P}),\\
&I_{(11)} = \mathrm{Tr}(\overset{-1}{C}{}^K_{\cdot L}\overset{-1}{C}{}^L_{\cdot P}\hat{M}^P_{\cdot Q})
\end{aligned}\right\} \tag{3.21}$$

and writing

$$e = \tilde{e}(I_{(\beta)}), \qquad \alpha^{(\beta)} \equiv \frac{\partial \tilde{e}}{\partial I_{(\beta)}}, \qquad (\beta) = 1, 2, \ldots, 11, \tag{3.22}$$

we get from equations (3.22), (3.21), (3.14) and (3.15)

$$\begin{aligned}
{}^{\mathrm{R}}t^{\beta\alpha} = \rho\Bigg\{-2\Bigg(\sum_{(\beta)=1,2,3,5,6,10,11} \alpha^{(\beta)}\frac{\partial I_{(\beta)}}{\partial \overset{-1}{C}{}^{MN}}\Bigg)X^{M,\alpha} + \Bigg(\sum_{(\beta)=4}^{6} \alpha^{(\beta)}\frac{\partial I_{(\beta)}}{\partial \hat{M}^N}\Bigg)\tilde{\mathscr{M}}^{\alpha}\\
+\Bigg(\sum_{(\beta)=7}^{11} \alpha^{(\beta)}\frac{\partial I_{(\beta)}}{\partial \hat{M}^{NM}}\Bigg)\tilde{\mathscr{M}}^{\alpha}_{\cdot;\mu}X^{M,\mu}\Bigg\}X^{N,\beta}
\end{aligned} \tag{3.23}$$

$${}^{\mathrm{R}}M^{\gamma\beta\alpha} = \rho\Bigg(\sum_{(\beta)=7}^{11} \alpha^{(\beta)}\frac{\partial I_{(\beta)}}{\partial \hat{M}^{MN}}\Bigg)\tilde{\mathscr{M}}^{[\gamma}X^{M,\beta]}X^{N,\mu} \tag{3.24}$$

in which the $\alpha^{(\beta)}$ are scalar functions of the invariants $I_{(\beta)}$. If one defines the following spatial PU measures of deformations:

$$\left.\begin{aligned}
&c_{\alpha\beta} \equiv G_{KL}X^K_{\cdot,\alpha}X^L_{\cdot,\beta} = c_{\beta\alpha}, \qquad c_{\alpha\beta}u^{\alpha} = 0,\\
&\mathfrak{M}_{\alpha\beta} \equiv P_{\alpha\mu}\tilde{\mathscr{M}}^{\mu}_{\cdot;\nu}P^{\nu}_{\cdot\beta}, \qquad \mathfrak{M}_{\alpha\beta}u^{\alpha} = \mathfrak{M}_{\alpha\beta}u^{\beta} = 0,
\end{aligned}\right\} \tag{3.25}$$

of which the former is the relativistic Cauchy strain tensor (cf Maugin 1971a), then, upon using equations (3.13) through (3.25), we can show that spatial forms of equations (3.23) and (3.24) are

$$
\begin{aligned}
{}^{\mathrm{R}}t^{\beta\alpha} = & -2\rho\{(\alpha^{(1)} - \hat{\boldsymbol{M}}^2\alpha^{(5)})c^{\alpha\beta} + (\alpha^{(2)} - \alpha^{(6)}\hat{\boldsymbol{M}}^2)c^{\alpha}_{\cdot\gamma}c^{\gamma\beta} + \alpha^{(3)}c^{\alpha}_{\cdot\gamma}c^{\gamma}_{\cdot\lambda}c^{\lambda\beta} + \alpha^{(5)}c^{\alpha}_{\cdot\gamma}c^{\cdot\beta}_{\lambda}\tilde{\mathcal{M}}^{\gamma}\tilde{\mathcal{M}}^{\lambda} \\
& + \alpha^{(10)}c^{\alpha}_{\cdot\mu}c^{\beta}_{\cdot\lambda}\mathfrak{M}^{\lambda\mu} + \tfrac{1}{2}\alpha^{(6)}\tilde{\mathcal{M}}^{\gamma}\tilde{\mathcal{M}}^{\nu}(c^{\alpha}_{\cdot\nu}c^{\beta}_{\cdot\lambda}c^{\lambda}_{\cdot\gamma} + c^{\alpha}_{\cdot\lambda}c^{\lambda}_{\cdot\gamma}c^{\cdot\beta}_{\nu}) \\
& + \alpha^{(11)}(c^{\alpha\rho}c^{\beta}_{\cdot\lambda}c^{\lambda\nu}\mathfrak{M}_{\nu\rho} + c^{\alpha\gamma}c_{\gamma\nu}c^{\cdot\beta}_{\lambda}\mathfrak{M}^{\lambda\nu})\} \\
& + \rho[\{(\alpha^{(4)} - 2I_{(1)}\alpha^{(5)} - 2I_{(2)}\alpha^{(6)})c^{\beta}_{\cdot\lambda} + \alpha^{(5)}c^{\beta\gamma}c_{\gamma\lambda} + \alpha^{(6)}c^{\beta\nu}c_{\nu\gamma}c^{\gamma}_{\cdot\lambda}\}]\tilde{\mathcal{M}}^{\alpha}\tilde{\mathcal{M}}^{\lambda} \\
& + \rho[\{\alpha^{(7)}c^{\mu\beta} + \alpha^{(8)}c^{\beta}_{\cdot\lambda}c^{\mu}_{\cdot\nu}\mathfrak{M}^{\lambda\nu} + \alpha^{(9)}c^{\mu\delta}c^{\cdot\beta}_{\gamma}c^{\cdot\lambda}_{\nu}\mathfrak{M}_{\lambda\delta} + \alpha^{(10)}c^{\mu}_{\cdot\lambda}c^{\lambda\beta} \\
& + \alpha^{(11)}c^{\beta}_{\cdot\lambda}c^{\lambda\gamma}c^{\cdot\mu}_{\gamma}\}\mathfrak{M}^{\alpha}_{\cdot\mu}]
\end{aligned}
\tag{3.26}
$$

$$
{}^{\mathrm{R}}M^{\gamma\beta\mu} = \rho\tilde{\mathcal{M}}^{[\gamma}(\alpha^{(7)}c^{\beta]\mu} + \alpha^{(8)}c^{\beta]}_{\cdot\lambda}\mathfrak{M}^{\lambda}_{\cdot\delta}c^{\delta\mu} + \alpha^{(9)}c^{\beta]}_{\cdot\lambda}c^{\alpha}_{\cdot\delta}\mathfrak{M}^{\lambda\delta}\mathfrak{M}_{\alpha\nu}c^{\nu\mu} + \alpha^{(10)}c^{\beta]}_{\cdot\lambda}c^{\lambda\mu} + \alpha^{(11)}c^{\beta]}_{\cdot\lambda}c^{\lambda}_{\cdot\delta}c^{\delta\mu}).
\tag{3.27}
$$

Furthermore, if there is no magnetization, then after use of the Cayley–Hamilton theorem in order to express $\boldsymbol{c}^3$ as a function of $\boldsymbol{c}$, $\boldsymbol{c}^2$ and $P^{\alpha\beta}$, the first three terms in the first brackets of equation (3.26) yield the Murnaghan form for the constitutive stress equation of purely nonlinear elastic solids (cf equation (3.32) in Grot and Eringen 1966).

It is of interest to note the different effects represented in equations (3.26) and (3.27). In order to do this, it is not necessary to linearize these constitutive equations by considering an expansion of the energy function (3.10) as a function of its arguments (as it was done in the simpler non-Lorentz invariant theory by Maugin and Eringin 1972b). We only have to examine the list of invariants (3.21) to judge the prominent influence of the terms in factor of each $\alpha^{(\beta)}$ in equation (3.26). The invariants that contain only $\overset{-1}{\boldsymbol{C}}$ will lead to the now classical (nonlinear) *elastic* effects. These are the invariants $I_{(1)}$, $I_{(2)}$ and $I_{(3)}$. That which involves only $\hat{M}^L$ (or $\boldsymbol{\Sigma}$), that is, $I_{(4)}$, yields the effect of *magnetic anisotropy*. Those which involve both $\hat{M}^L$ and the deformation tensor $\overset{-1}{C}{}^{KL}$, that is, $I_{(5)}$ and $I_{(6)}$, will yield a prominent magnetoelastic effect known as *magnetostriction*. Finally, those that contain $\hat{M}^{LN}$, that is, $I_{(7)}$ through $I_{(11)}$, describe the effect of *Heisenberg's exchange forces* which represent the interaction between neighbouring electronic spins. We thus have a physical significance given to each term in the factor of each $\alpha^{(\beta)}$ (one $\alpha^{(\beta)}$ corresponding to the same numbered $I_{(\beta)}$). However most effects are here mixed up in an intricate way because of the nonlinearity. For a linear theory, one would consider the limited expansion

$$
\tilde{e} = \tfrac{1}{2}L_{KLMN}\overset{-1}{C}{}^{KL}\overset{-1}{C}{}^{MN} + \tfrac{1}{2}B_{KL}\hat{M}^K\hat{M}^L + \tfrac{1}{2}\lambda_{KLMN}\overset{-1}{C}{}^{KL}\hat{M}^M\hat{M}^N + \tfrac{1}{2}A_{KLMN}\hat{M}^{KL}\hat{M}^{MN}
\tag{3.28}
$$

in which the constant coefficients L_{KLMN}, B_{KL}, λ_{KLMN} and A_{KLMN} satisfy obvious symmetry relations. The different contributions are here easily recognized as being the *elastic* energy, the *magnetic anisotropy* energy, the *magnetoelastic* energy and the *exchange* energy respectively. For isotropy, the coefficients introduced assume their isotropic values(combinations of Kronecker symbols δ_{KL}). For such developments, we refer the reader to the classical theory of micromagnetism such as developed by Maugin and Eringen (1972b, see also Maugin 1972e).

Finally, as is clearly seen from equation (3.27) ${}^{\mathrm{R}}M^{\gamma\beta\mu}$ results mainly from the action of exchange forces; in the linear theory constructed with the energy (3.28) it would depend only on these forces.

4. Constitutive equations for isotropic dissipative solids

4.1. Preliminaries

Once constitutive equations have been derived for ${}^{\mathrm{R}}t^{\beta\alpha}$ and ${}^{\mathrm{R}}M^{\alpha\beta\gamma}$ according to equations (3.8)–(3.9) or (3.14)–(3.15), the Clausius–Duhem inequality (2.5) reduces to the dissipation inequality (cf equations (II-3.31) and (II-3.38)):

$$\Phi \equiv \mathscr{E}_{\gamma} j^{\gamma} - \frac{1}{\theta}\hat{q}^{\beta}\overset{*}{\theta}_{\beta} + {}^{\mathrm{D}}t^{(\beta\alpha)}\sigma_{\alpha\beta} + {}^{\mathrm{D}}t^{[\beta\alpha]}(\omega_{\alpha\beta} - \Omega_{\alpha\beta}) + {}^{\mathrm{D}}M^{\beta\alpha\gamma}\mathscr{A}_{\alpha\beta\gamma} \geqslant 0. \tag{4.1}$$

We define the *relative angular velocity tensor* of the magnetization by

$$\nu_{\alpha\beta} \equiv \omega_{\alpha\beta} - \Omega_{\alpha\beta}, \qquad \nu_{\alpha\beta} = -\nu_{\beta\alpha}, \qquad \nu_{\alpha\beta}u^{\alpha} = 0. \tag{4.2}$$

Since ${}^{\mathrm{D}}M^{\beta\alpha\gamma}$ and $\mathscr{A}_{\alpha\beta\gamma}$ are skewsymmetric in their two first indices (cf equations (2.15)), we can introduce their duals in M^4 in a unique way. That is,

$$\overset{*}{M}{}_{\beta}^{\cdot\gamma} \equiv \frac{1}{2\mathrm{i}c}\epsilon_{\beta\alpha\mu\nu}\,{}^{\mathrm{D}}M^{\alpha\mu\gamma}u^{\nu}, \qquad \overset{*}{M}{}_{\beta}^{\cdot\gamma}u^{\beta} = \overset{*}{M}{}_{\beta}^{\cdot\gamma}u_{\gamma} = 0, \tag{4.3}$$

$$\overset{*}{\mathscr{A}}{}^{\lambda}_{\cdot\gamma} \equiv \frac{1}{2\mathrm{i}c}\epsilon^{\lambda\beta\alpha\sigma}\mathscr{A}_{\beta\alpha\gamma}u_{\sigma}, \qquad \overset{*}{\mathscr{A}}{}^{\lambda}_{\cdot\gamma}u_{\lambda} = \overset{*}{\mathscr{A}}{}^{\lambda}_{\cdot\gamma}u^{\gamma} = 0. \tag{4.4}$$

Reciprocally,

$${}^{\mathrm{D}}M^{\beta\alpha\gamma} = \frac{1}{\mathrm{i}c}\epsilon^{\beta\alpha\rho\sigma}\overset{*}{M}{}_{\rho}^{\cdot\gamma}u_{\sigma}, \qquad \mathscr{A}_{\beta\alpha\gamma} = \frac{1}{\mathrm{i}c}\epsilon_{\beta\alpha\mu\sigma}\overset{*}{\mathscr{A}}{}^{\mu}_{\cdot\gamma}u^{\sigma}. \tag{4.5}$$

Then using equations (4.5) and (2.13), we transform the last term in Φ, and taking account of equation (4.2), we write the dissipation density Φ as

$$\Phi = \mathscr{E}_{\gamma} j^{\gamma} - \frac{1}{\theta}\hat{q}^{\beta}\overset{*}{\theta}_{\beta} + {}^{\mathrm{D}}t^{(\beta\alpha)}\sigma_{\alpha\beta} + {}^{\mathrm{D}}t^{[\beta\alpha]}\nu_{\alpha\beta} - 2\overset{*}{M}{}^{\mu\gamma}\overset{*}{\mathscr{A}}_{\mu\gamma}. \tag{4.6}$$

The present problem consists in constructing constitutive equations for j^{γ}, $\hat{q}^{\beta}$, ${}^{\mathrm{D}}t^{(\beta\alpha)}$, ${}^{\mathrm{D}}t^{[\beta\alpha]}$ and $\overset{*}{M}{}^{\mu\gamma}$. One may assume that $\mathscr{E}_{\gamma}$, $\overset{*}{\theta}_{\beta}$, $\sigma_{\alpha\beta}$, $\nu_{\alpha\beta}$ and $\overset{*}{\mathscr{A}}_{\mu\gamma}$ are the thermodynamical affinities to which are associated the corresponding generalized fluxes j^{γ}, $-\hat{q}^{\beta}/\theta$, ${}^{\mathrm{D}}t^{(\beta\alpha)}$, ${}^{\mathrm{D}}t^{[\beta\alpha]}$ and $-2\overset{*}{M}{}^{\mu\gamma}$. We shall denote by $\dot{\alpha}_{(\beta)}$, $(\beta) = 1, 2, \ldots, 24$, the indexed series of independent components of the generalized affinities† and by $\chi_{(\beta)}$, $(\beta) = 1, 2, \ldots, 24$, the indexed series of independent components of the generalized fluxes (or *irreversible forces* in the language of Ziegler 1963). From these considerations, *linear* constitutive equations could be immediately constructed for dissipative phenomena by means of Onsager's theory. However a recent trend in continuum physics has been to generalize the concept of potential to *nonlinear* dissipative phenomena (this potential being subject to the convexity condition provided by inequality (4.1)), thus allowing a degree of generality similar to that provided by the potential ψ^* or e for recoverable phenomena. Among the most notable attempts toward this direction is that of Ziegler (1963) to which Moreau (1970) has brought more mathematical rigour (cf Germain 1973). We have already used

† All tensor fields that appear in Φ are PU. Thus, although they are expressed in covariant four-dimensional formalism, they have exactly the same number of independent components as the corresponding three-dimensional entities, for example, each of $\mathscr{E}_{\gamma}$, $\overset{*}{\theta}_{\beta}$ and $\nu_{\alpha\beta}$ has three independent components, $\sigma_{\alpha\beta}$ has six, and $\overset{*}{\mathscr{A}}_{\mu\gamma}$ has nine. The same is true of the corresponding fluxes.

this method for the treatment of dissipative phenomena in the classical theory of micromagnetics (Maugin 1972d). Here we shall recall without derivation the more modern approach envisaged by the author (Maugin 1973 preprint).

First, in general all $\chi_{(\beta)}$ depend on all $\dot{\alpha}_{(\beta)}$ after the axiom of equipresence (Eringen 1962, chap 5). Next, the $\chi_{(\beta)}$ may also depend on the arguments which appeared in ψ^* or e, that is, the arguments used to define the recoverable phenomena. Let A denote the set of these arguments. Hence

$$\chi_{(\beta)} = \tilde{\chi}_{(\beta)}(A, \dot{\alpha}_{(\gamma)}), \qquad (\beta), (\gamma) = 1, 2, \ldots, 24 \tag{4.7}$$

with, symbolically (cf equation (4.1)),

$$\Phi = \sum_{(\beta)} \chi_{(\beta)} \dot{\alpha}_{(\beta)} \geqslant 0. \tag{4.8}$$

Now, if the $\chi_{(\beta)}$ are assumed to be functions of class C^1 with respect to the arguments $\dot{\alpha}_{(\beta)}$, the inequality (4.8) clearly implies for a given (β) that

$$\tilde{\chi}_{(\beta)}(A, \dot{\alpha}_{(\gamma) \neq (\beta)}, \dot{\alpha}_{(\beta)} = 0) = 0. \tag{4.9}$$

The general formulation proposed by the author (Maugin 1973 preprint) goes as follows. Define the total dissipation or entropy production $P_\eta(\mathring{\mathscr{B}})$ in an open region $(\mathring{\mathscr{B}})$ of M^4, that is,

$$P_\eta(\mathring{\mathscr{B}}) = \int_{(\mathring{\mathscr{B}})} \Phi \, \mathrm{d}^4 v. \tag{4.10}$$

We then have the *extremum theorem*†:

If there exists a continuous functional $\mathscr{F}[A, \dot{\alpha}_{(\beta)}]$ convex with respect to the $\dot{\alpha}_{(\beta)}$ for all A fixed and which possesses partial Fréchet derivatives with respect to $\dot{\alpha}_{(\beta)}$ that vanish for $\dot{\alpha}_{(\beta)} = 0$, that is, if $\mathscr{F}$ is stationary at $\dot{\alpha}_{(\beta)} = 0$, then setting

$$P_\eta(\mathring{\mathscr{B}}) = \delta_{\mathrm{T}} \mathscr{F}[A, \dot{\alpha}_{(\beta)}], \qquad A \text{ fixed} \tag{4.11}$$

where $\delta_{\mathrm{T}}\mathscr{F}$ indicates the total Fréchet derivative of $\mathscr{F}$, the second principle of thermodynamics is automatically satisfied in global form (for $\mathring{\mathscr{B}}$), for

$$P_\eta(\mathring{\mathscr{B}}) \geqslant 0 \qquad \text{with} \qquad P_\eta(\mathring{\mathscr{B}})\big|_{\dot{\alpha}_{(\beta)} = 0} = 0. \tag{4.12}$$

Further we have the following representation and lower and upper bounds

$$\begin{aligned} 0 \leqslant \delta_{\mathrm{T}} \mathscr{F}[A, \dot{\alpha}_{(\beta)}; \dot{\alpha}_{(\beta)}] &= \sum_{(\beta)} \delta_{\dot{\alpha}_{(\beta)}} \mathscr{F}[A, \dot{\alpha}_{(\gamma)}; \dot{\alpha}_{(\beta)}] = \sum_{(\beta)} \nabla_{\dot{\alpha}_{(\beta)}} \mathscr{F}[A, \dot{\alpha}_{(\gamma)}] \dot{\alpha}_{(\beta)} \\ &= \int_{(\mathring{\mathscr{B}})} \sum_{(\beta)} \tilde{\chi}_{(\beta)}(A, \dot{\alpha}_{(\gamma)}) \dot{\alpha}_{(\beta)} \, \mathrm{d}^4 v \leqslant \mathscr{F}[A, 2\dot{\alpha}_{(\beta)}] - \mathscr{F}[A, \dot{\alpha}_{(\beta)}] \end{aligned} \tag{4.13}$$

in which $\delta_{\dot{\alpha}_{(\beta)}} \mathscr{F}$ indicates the partial Fréchet derivative with respect to $\dot{\alpha}_{(\beta)}$ and $\nabla_{\dot{\alpha}_{(\beta)}} \mathscr{F}$ is the partial gradient of $\mathscr{F}$ with respect to $\dot{\alpha}_{(\beta)}$.

Clearly the $\chi_{(\beta)}$ are derivable from the potential $\mathscr{F}$. The particular case of interest is the following. Let $\tilde{\Phi}(A, \dot{\alpha}_{(\beta)})$ be a convex *function* with respect to $\dot{\alpha}_{(\beta)}$ for all A, at least of class C^2 with respect to the $\dot{\alpha}_{(\beta)}$ and such that

$$\left.\frac{\partial \tilde{\Phi}}{\partial \dot{\alpha}_{(\beta)}}\right|_{\dot{\alpha}_{(\beta)} = 0} = 0.$$

† A, $\dot{\alpha}_{(\beta)}$ and $\mathscr{F}$ have values in well-defined vectorial spaces of functions that we do not specify here. The symbolism $\mathscr{F}[A, B; C]$ means that $\mathscr{F}$ is a general functional of A and B and a *continuous linear* functional of C. Compare Rall (1971) for elements of nonlinear functional analysis.

Then the functional

$$\mathscr{F}[A, \dot{\alpha}_{(\beta)}] = \int_{(\mathring{\mathscr{B}})} \tilde{\Phi}(A, \dot{\alpha}_{(\beta)})\, \mathrm{d}^4 v$$

verifies the hypotheses of the extremum theorem enunciated above, and

$$\chi_{(\beta)} = \frac{\partial \tilde{\Phi}(A, \dot{\alpha}_{(\gamma)})}{\partial \dot{\alpha}_{(\beta)}}. \tag{4.14}$$

Further, $\tilde{\Phi}$ being assumed of class C^2, we get from equation (4.14) the reciprocity relations

$$\frac{\partial \chi_{(\beta)}}{\partial \dot{\alpha}_{(\gamma)}} = \frac{\partial \chi_{(\gamma)}}{\partial \dot{\alpha}_{(\beta)}}, \qquad (\gamma), (\beta) \text{ fixed.} \tag{4.15}$$

If $\tilde{\Phi}$ is homogeneous of degree two in $\dot{\alpha}_{(\beta)}$, then

$$\Phi = \sum_{(\beta)} \frac{\partial \tilde{\Phi}}{\partial \dot{\alpha}_{(\beta)}} \dot{\alpha}_{(\beta)} = 2\tilde{\Phi},$$

and the relations (4.15) degenerate into the classical Onsager reciprocity relations.

Hence the $\chi_{(\beta)} = (j^\gamma, -\hat{q}^\beta/\theta, {}^{\mathrm{D}}t^{(\beta\alpha)}, {}^{\mathrm{D}}t^{[\beta\alpha]}, -2\overset{*}{M}{}^{\mu\gamma})$ in general are derivable from a *dissipation potential* $\tilde{\Phi}$.

4.2. *Approximate constitutive equations*

The degree of generality obtained from the preceding formulation is pretty large but it also leads to complex expressions, for we should consider, for *isotropic media*, a representation of the dissipation potential

$$\tilde{\Phi} = \tilde{\Phi}(A, \sigma_{\alpha\beta}, \nu_{\alpha\beta}, \overset{*}{\mathscr{A}}_{\mu\gamma}, \mathscr{E}_\gamma, \overset{*}{\theta}_\beta) \tag{4.16}$$

in which

$$A = (c_{\alpha\beta}, \tilde{\mathscr{M}}_\alpha, \mathfrak{M}_{\alpha\beta}, \theta) \tag{4.17}$$

as an *isotropic* function of its PU arguments. This is in general possible but we should get extremely complicated expressions as one can already judge from the lengthy unmanageable expressions given, for instance, by Grot and Eringen (1966) and Eringen (1970) when the number of arguments is less than in the present case. Since the present theory is already much complicated by itself and we desire to put in evidence the leading effects, we shall content ourselves with the following 'natural' (cf appendix 1) dissipation potential

$$\begin{aligned}\tilde{\Phi} = \frac{1}{2}\Big(& \lambda(\mathrm{Tr}\,\boldsymbol{\sigma})^2 + 2\mu\,\mathrm{Tr}\,\boldsymbol{\sigma}^2 + \bar{\xi}\,\mathrm{Tr}\,\mathbf{v}^2 + \sigma\mathscr{E}^2 + \frac{\chi}{\theta}\overset{*}{\theta}{}^2\Big) \\ & - \{\nu(\mathrm{Tr}\,\overset{*}{\mathscr{A}})^2 + \zeta\,\mathrm{Tr}(\overset{*}{\mathscr{A}}\overset{*}{\mathscr{A}}{}^{\mathrm{T}}) + \xi\,\mathrm{Tr}\,\overset{*}{\mathscr{A}}{}^2\}\end{aligned} \tag{4.18}$$

in which the scalar coefficients λ, μ, $\bar{\xi}$, σ, χ, ν, ζ and ξ are considered to be scalar valued functions of $A = (\rho, \theta, \tilde{\mathscr{M}}_{\mathrm{S}}^2 \equiv \tilde{\mathscr{M}}^\alpha \tilde{\mathscr{M}}_\alpha)$. Here we have set (T = transposed)

$$\begin{gathered}\mathrm{Tr}\,\boldsymbol{\sigma}^2 \equiv \sigma^{\alpha\beta}\sigma_{\beta\alpha}, \qquad \mathrm{Tr}\,\mathbf{v}^2 \equiv \nu^{\alpha\beta}\nu_{\beta\alpha}, \qquad \mathscr{E}^2 \equiv \mathscr{E}^\gamma\mathscr{E}_\gamma, \qquad \overset{*}{\boldsymbol{\theta}}{}^2 \equiv P^{\alpha\beta}\overset{*}{\theta}_\alpha\overset{*}{\theta}_\beta, \\ \mathrm{Tr}(\overset{*}{\mathscr{A}}\overset{*}{\mathscr{A}}{}^{\mathrm{T}}) \equiv \overset{*}{\mathscr{A}}{}^{\mu\gamma}\overset{*}{\mathscr{A}}_{\mu\gamma}, \qquad \mathrm{Tr}\,\overset{*}{\mathscr{A}}{}^2 \equiv \overset{*}{\mathscr{A}}{}^{\mu\gamma}\overset{*}{\mathscr{A}}_{\gamma\mu}.\end{gathered} \tag{4.19}$$

The potential $\tilde{\Phi}$ defined by (4.18) verifies the stationary hypothesis. It is homogeneous of degree two in its arguments $\dot{\alpha}_{(\beta)}$. Clearly it is valid for *isotropic media* (cf appendix 1). It satisfies the convexity hypothesis if and only if the quadratic form (4.18) so defined is positive definite. This imposes some restrictions on the values of the scalar coefficients introduced. The necessary and sufficient conditions are established as follows. We introduce the symmetric and skewsymmetric parts $\overset{*}{\mathscr{A}}_{(\cdot)}$ and $\overset{*}{\mathscr{A}}_{[\cdot]}$ of $\overset{*}{\mathscr{A}}$ and the deviatoric parts ${}_{\mathrm{d}}\boldsymbol{\sigma}$ and ${}_{\mathrm{d}}\overset{*}{\mathscr{A}}$ of $\boldsymbol{\sigma}$ and $\overset{*}{\mathscr{A}}_{(\cdot)}$ by

$$\overset{*}{\mathscr{A}}_{(\mu\gamma)} \equiv \tfrac{1}{2}(\overset{*}{\mathscr{A}}_{\mu\gamma}+\overset{*}{\mathscr{A}}_{\gamma\mu}), \qquad \overset{*}{\mathscr{A}}_{[\mu\gamma]} = \tfrac{1}{2}(\overset{*}{\mathscr{A}}_{\mu\gamma}-\overset{*}{\mathscr{A}}_{\gamma\mu}),$$
$${}_{\mathrm{d}}\sigma_{\alpha\beta} \equiv \sigma_{\alpha\beta}-\tfrac{1}{3}(\mathrm{Tr}\,\boldsymbol{\sigma})P_{\alpha\beta}, \qquad {}_{\mathrm{d}}\overset{*}{\mathscr{A}}_{\mu\gamma} \equiv \overset{*}{\mathscr{A}}_{(\mu\gamma)}-\tfrac{1}{3}(\mathrm{Tr}\,\overset{*}{\mathscr{A}})P_{\mu\gamma}. \tag{4.20}$$

Note that $\mathrm{Tr}\,\boldsymbol{P} = P^{\alpha}_{\cdot\alpha} = 3$ (cf part I) so that

$$\mathrm{Tr}\,\overset{*}{\mathscr{A}}{}^{2}_{(\cdot)} = \mathrm{Tr}\,{}_{\mathrm{d}}\overset{*}{\mathscr{A}}{}^{2}+\tfrac{1}{3}(\mathrm{Tr}\,\overset{*}{\mathscr{A}})^{2}, \tag{4.21}$$

A similar formula holds for $\boldsymbol{\sigma}$ and ${}_{\mathrm{d}}\boldsymbol{\sigma}$. Carrying (4.20) and (4.21) into equation (4.18), we obtain

$$\tilde{\Phi} = \tfrac{1}{2}\left(\tfrac{1}{3}(3\lambda+2\mu)(\mathrm{Tr}\,\boldsymbol{\sigma})^{2}+2\mu\,\mathrm{Tr}\,{}_{\mathrm{d}}\boldsymbol{\sigma}^{2}+\sigma\mathscr{E}^{2}+\frac{\chi}{\theta}\overset{*}{\boldsymbol{\theta}}{}^{2}+\bar{\xi}\,\mathrm{Tr}\,\mathbf{v}^{2}\right)$$
$$-\{\tfrac{1}{3}(3\nu+\zeta+\xi)(\mathrm{Tr}\,\overset{*}{\mathscr{A}})^{2}+(\zeta-\xi)\,\mathrm{Tr}(\overset{*}{\mathscr{A}}_{[\cdot]}\cdot\overset{*}{\mathscr{A}}{}^{\mathrm{T}}_{[\cdot]})+(\zeta+\xi)(\mathrm{Tr}\,{}_{\mathrm{d}}\overset{*}{\mathscr{A}}{}^{2})\}. \tag{4.22}$$

Then for $\tilde{\Phi}$ to be non-negative, it is *sufficient* that

$$3\lambda+2\mu \geqslant 0, \qquad \mu \geqslant 0, \qquad \sigma \geqslant 0, \qquad \chi \geqslant 0, \qquad \bar{\xi} \geqslant 0,$$
$$3\nu+\zeta+\xi \leqslant 0, \qquad \xi+\zeta \leqslant 0 \leqslant \xi-\zeta. \tag{4.23}$$

Using a method similar to that used by Eringen (1968, p 694)†, one may prove that the conditions (4.23) are also *necessary*.

It is now straightforward to give the constitutive equations that correspond to the dissipation potential (4.22). Applying equation (4.14), we immediately get

$${}^{\mathrm{D}}t^{(\beta\alpha)} = \lambda(\rho,\theta,\tilde{\mathscr{M}}{}^{2}_{\mathrm{S}})\sigma^{\gamma}_{\cdot\gamma}P^{\beta\alpha}+2\mu(\rho,\theta,\tilde{\mathscr{M}}{}^{2}_{\mathrm{S}})\sigma^{\alpha\beta}, \tag{4.24}$$

$${}^{\mathrm{D}}t^{[\beta\alpha]} = \bar{\xi}(\rho,\theta,\tilde{\mathscr{M}}{}^{2}_{\mathrm{S}})v^{\beta\alpha}, \tag{4.25}$$

$$j^{\gamma} = \sigma(\rho,\theta,\tilde{\mathscr{M}}{}^{2}_{\mathrm{S}})\mathscr{E}^{\gamma}, \tag{4.26}$$

$$\hat{q}^{\beta} = -\chi(\rho,\theta,\tilde{\mathscr{M}}{}^{2}_{\mathrm{S}})P^{\beta\alpha}\overset{*}{\theta}_{\alpha}, \tag{4.27}$$

$$\overset{*}{M}{}^{\mu\gamma} = \nu(\rho,\theta,\tilde{\mathscr{M}}{}^{2}_{\mathrm{S}})\overset{*}{\mathscr{A}}{}^{\sigma}_{\cdot\sigma}P^{\mu\gamma}+a(\rho,\theta,\tilde{\mathscr{M}}{}^{2}_{\mathrm{S}})\overset{*}{\mathscr{A}}{}^{(\mu\gamma)}+b(\rho,\theta,\tilde{\mathscr{M}}{}^{2}_{\mathrm{S}})\overset{*}{\mathscr{A}}{}^{[\mu\gamma]} \tag{4.28}$$

where $a \equiv \zeta+\xi$ and $b \equiv \zeta-\xi$. Finally, from equations (4.28), (4.5), (4.4) and (2.14), we obtain

$$2\,{}^{\mathrm{D}}M^{\beta\alpha\gamma} = \nu H^{[\alpha\beta\gamma]}+2\zeta\mathscr{A}^{\alpha\beta\gamma}+\xi L^{\alpha\beta\gamma} \tag{4.29}$$

in which we have defined the PU quantities

$$H^{[\alpha\beta\gamma]} \equiv \frac{1}{c^{2}}\epsilon^{\alpha\beta\gamma\sigma}\epsilon^{\lambda\mu\nu\rho}\mathscr{A}_{\mu\nu\lambda}u_{\sigma}u_{\rho},$$

† Equation (4.22) and equation (21.4) of Eringen (1968)—who deals with micropolar elasticity—have formally the same structure, we thus refer the reader to Eringen for the proof.

$$L^{\alpha\beta\gamma} \equiv \frac{1}{c^2}\epsilon^{\alpha\beta\rho\sigma}\epsilon^{\gamma\mu\nu\xi}\mathscr{A}_{\mu\nu\rho}u_\sigma u_\xi,$$

of which the former is obviously completely antisymmetric.

Equations (4.24) through (4.27) and equation (4.29) form a set of possible constitutive equations for dissipative phenomena in *isotropic* media with electronic spins. A possible interpretation of these different phenomena is discussed in the next section.

4.3. *Interpretation of some dissipative phenomena*

4.3.1. Electrical conduction. Clearly the scalar σ represents the electrical conductivity and equation (4.26) is the four-dimensional version of Ohm's law. For infinite electrical conductivity, we should take $\mathscr{E}^\gamma = 0$. That is, neglecting the relativistic effects and using the classical three-dimensional notation for the electric, magnetic induction, and velocity fields (cf Maugin 1972f),

$$\boldsymbol{E}+\frac{1}{c}\boldsymbol{v}\times\boldsymbol{B} = 0, \qquad \boldsymbol{v}\,.\,\boldsymbol{E} = 0. \tag{4.30}$$

The first of these is classical Ohm's law of perfect magnetohydrodynamics. The second of equations (4.30) asserts that the Joule term vanishes for pure electric convection.

4.3.2. Viscous stresses. Equations (4.24) represent constitutive equations for relativistic newtonian compressible fluids (cf Maugin 1971b). The coefficients λ and μ are scalar viscosities. Together with the constitutive equations (3.26) which can be linearized with an energy density given by equation (3.28) in a manner similar to that used in the classical theory (cf Maugin and Eringen 1972b), equations (4.24) provide constitutive equations for a material of the *viscoelastic* type. This dual mechanical behaviour coupled with the magnetic properties described in the present theory—rotation of the magnetization field, interactions between magnetization field and matter, interactions between spins—provides an interesting possibility, namely the phenomenological description of what occurs in liquid crystals. These peculiar media present either a liquid or an elastic behaviour according to the circumstances. Moreover, placed in a magnetic field, they exhibit several magnetic effects (orientation of the spins). It is believed that their magnetic properties should be accurately described by the equation of evolution of the magnetization obtained here. It is known (cf Lee and Eringen 1971) that the stress tensor is not symmetric in such media because of the existence of spin, applied couples and couple stresses. One of course does not need a relativistic theory, but the three-dimensional expressions valid in the limit $c \to \infty$ can easily be deduced from the present formulation by applying the method outlined in Maugin and Eringen (1972c). Also, it is to be remarked that liquid crystals exhibit a strong asymmetry in their mechanical properties. They are mainly transversely isotropic, that is, they are endowed with preferred directions. The degree of symmetry considered in the present work—isotropy—is then too large. However, at least in a linear theory based on an energy density (3.28), it is possible to develop constitutive equations for transverse isotropy. The possibility pointed out here remains to be tested. At present this is outside the scope of this paper.

Finally, while we know of only two mechanical behaviours which correspond to recoverable thermodynamical phenomena—pure elasticity and perfect fluidity—the class of constitutive assumptions for dissipative mechanical phenomena is much larger. Equations (4.24) offer but a very simple possibility. In relativistic continuum mechanics, more complicated ones have been envisaged elsewhere, for example, Kelvin–Voigt viscoelastic materials in Maugin (1973d, e).

4.3.3. Interpretation of ${}^{\mathrm{D}}t^{[\beta\alpha]}$. The 'dissipative' constitutive equations (4.25) are the more interesting ones among those given in § 4.2, for they yield effects peculiar to the theory. First it is to be noticed that ${}^{\mathrm{D}}t^{[\beta\alpha]}$ is—in isotropic materials—proportional to the relative angular velocity (cf equation (4.2)) that vanishes whenever the magnetization locally rotates, in an inertial frame, at the same rate as does the deformable matter, that is, when the magnetization is *frozen* in the material. If it is so —$\nu^{\beta\alpha} = 0$—the only consequence is that ${}^{\mathrm{D}}t^{[\beta\alpha]} = 0$†. Furthermore, if ${}^{\mathrm{D}}\dot{M}^{\mu\gamma} = 0$ or, equivalently, ${}^{\mathrm{D}}M^{\beta\alpha\gamma} = 0$, then it is known that equation (2.3) can be transformed into other equivalent forms which represent a precessional motion of the magnetization. The result established in part II (see also appendix 2 hereafter) is that, in an inertial frame, the magnetization *precesses at a uniform angular velocity* proportional to an effective magnetic induction. The situation is somewhat similar to that of a top which, when acted upon by gravity, precesses at a uniform angular velocity in the absence of friction with air. Then we can infer that if $\nu^{\beta\alpha}$ differs from zero, hence ${}^{\mathrm{D}}t^{[\beta\alpha]} \neq 0$, the precessional motion of the magnetization will vary in time as a consequence of dissipative effects due to internal friction between the rotating magnetization and the deformable medium. If the medium were rigid, then $\nu^{\beta\alpha}$ would reduce to $\Omega^{\alpha\beta}$ and the effect would remain essentially unchanged. We shall put this leading effect in evidence by transforming equation (2.3). In order to do this, we need certain expressions for the angular velocity $\Omega^{\alpha\beta}$.

We start with equation (A.4) in which π^{μ} is defined by the second of (A.7). Multiplying both sides by $(1/\mathrm{i}c)\epsilon^{\alpha\lambda\gamma\rho}\tilde{\mathscr{M}}_{\lambda}u_{\rho}$ and using equations (2.14) and $u^{\alpha}u_{\alpha}+c^2 = 0$, we obtain

$$\pi^{\gamma} = \frac{1}{\tilde{\mathscr{M}}_{\mathrm{S}}^{2}}\left(\frac{1}{\mathrm{i}c}\epsilon^{\alpha\lambda\gamma\rho}\dot{\tilde{\mathscr{M}}}_{\alpha}\tilde{\mathscr{M}}_{\lambda}u_{\rho}+(\tilde{\mathscr{M}}_{\mu}\pi^{\mu})\tilde{\mathscr{M}}^{\gamma}\right), \tag{4.31}$$

or equivalently for the relativistic dual $\Omega_{\mu\nu}$ of π^{γ},

$$\Omega_{\mu\nu} = \frac{1}{\tilde{\mathscr{M}}_{\mathrm{S}}^{2}}\,\mathbf{P}\{2\dot{\tilde{\mathscr{M}}}_{[\mu}\tilde{\mathscr{M}}_{\nu]}+\tfrac{1}{2}S_{\nu\mu}(S_{\alpha\gamma}\Omega^{\gamma\alpha})\}, \tag{4.32}$$

hence the explicit form of equation (4.25). Now consider equation (2.3). It was shown in part II that, if ${}^{\mathrm{D}}t^{[\beta\alpha]}$ and ${}^{\mathrm{D}}M^{\alpha\beta\gamma}$ vanish, then this equation takes the form of equation (A.5) (appendix 2), that is, on account of equation (A.11),

$$\mathbf{P}\{\dot{S}^{\alpha\beta}\} = \mathbf{P}\{2\pi^{[\alpha}\tilde{\mathscr{M}}^{\beta]}\}, \qquad \pi^{\alpha} \equiv -\gamma\mathscr{B}^{\alpha}_{\mathrm{eff}}. \tag{4.33}$$

The first of these equations is essentially spatial. It reduces to the classical three-dimensional equation $\dot{\boldsymbol{\mu}} = \boldsymbol{\pi}\times\boldsymbol{\mu}$ ($\boldsymbol{\mu}$ is the magnetization three-dimensional vector per unit of mass) in which the three-dimensional angular velocity $\boldsymbol{\pi}$ varies from place to place in a magnetized medium, but is a constant in time at a certain spatial location. We assume that ${}^{\mathrm{D}}\dot{M}^{\mu\gamma} = 0$, that is, ${}^{\mathrm{D}}M^{\beta\alpha\gamma} = 0$, but ${}^{\mathrm{D}}t^{[\alpha\beta]} \neq 0$. Then after equation (2.3), we must add the quantity $(2\gamma/\rho){}^{\mathrm{D}}t^{[\alpha\beta]}$ to the right hand side of equation (4.33, part two). On account of the constitutive equation for ${}^{\mathrm{D}}t^{[\alpha\beta]}$, we obtain

$$\mathbf{P}\{\dot{S}^{\alpha\beta}\} = \mathbf{P}\left\{2\gamma\left(\tilde{\mathscr{M}}^{[\alpha}\mathscr{B}^{\beta]}_{\mathrm{eff}}+\frac{\alpha}{\tilde{\mathscr{M}}_{\mathrm{S}}}\tilde{\mathscr{M}}^{[\alpha}\dot{\tilde{\mathscr{M}}}^{\beta]}+\kappa S^{\alpha\beta}(\tfrac{1}{2}S_{\mu\nu}S^{\nu\mu})+\Psi\omega^{\alpha\beta}\right)\right\} \tag{4.34}$$

† In fact this is also true in anisotropic materials after the continuity condition (4.9).

in which we have set

$$\alpha(\rho, \theta, \tilde{\mathscr{M}}_{\mathrm{S}}) \equiv \frac{2\bar{\xi}(\rho, \theta, \tilde{\mathscr{M}}_{\mathrm{S}})}{\rho|\tilde{\mathscr{M}}_{\mathrm{S}}|}, \qquad \kappa(\rho, \theta, \tilde{\mathscr{M}}_{\mathrm{S}}) \equiv \frac{\alpha(\rho, \theta, \tilde{\mathscr{M}}_{\mathrm{S}})}{2|\tilde{\mathscr{M}}_{\mathrm{S}}|}$$

$$\Psi(\rho, \theta, \tilde{\mathscr{M}}_{\mathrm{S}}) \equiv \tfrac{1}{2}\alpha(\rho, \theta, \tilde{\mathscr{M}}_{\mathrm{S}})|\tilde{\mathscr{M}}_{\mathrm{S}}| \tag{4.35}$$

and $\mathscr{B}^{\alpha}_{\mathrm{eff}}$ is the effective magnetic induction defined in part II for recoverable phenomena. The last contribution within brackets in the expression (4.34) is due to the vorticity $\omega^{\alpha\beta}$. It vanishes whenever the medium considered is a rigid motionless solid or the matter motion is irrotational. With equation (4.34), it is no longer possible to define a time constant precession velocity for, even if we define

$$\tilde{\pi}^{\alpha} \equiv -\gamma\left(\mathscr{B}^{\alpha}_{\mathrm{eff}} + \frac{\alpha}{|\tilde{\mathscr{M}}_{\mathrm{S}}|}\dot{\tilde{\mathscr{M}}}^{\alpha}\right), \tag{4.36}$$

and discard the two last terms in the right hand side of equation (4.34), the angular velocity so defined will vary in time and the nonrelativistic three-dimensional limit of equation (2.3) will be

$$\dot{\boldsymbol{\mu}} = \tilde{\boldsymbol{\pi}} \times \boldsymbol{\mu} = \boldsymbol{\pi} \times \boldsymbol{\mu} - \frac{\alpha\gamma}{|\boldsymbol{\mu}_{\mathrm{S}}|}\dot{\boldsymbol{\mu}} \times \boldsymbol{\mu}. \tag{4.37}$$

This shows that the scalar α acts as a *damping constant* whose presence leads to a spiralling of the magnetization field. This was only forecast in part II for the relativistic case. However formulae have been obtained previously by the author (Maugin 1972d) for the classical three-dimensional theory of micromagnetics. The occurrence of the coefficient α materializes internal friction, a phenomenon which bears a statistical mechanics support. Thus equation (4.34) generalizes the equations of Maugin (1972d) to relativistic formalism with the associated relativistic effects. By way of consequence, the equations of Gilbert and Kelly (1955) and Landau and Lifshitz (1935) which are approximations (when the deformation is neglected in both cases and, further, with small damping in the second case) are included in the formulation given here.

In conclusion of this section, the existence of a 'dissipative' skewsymmetric part of the stress tensor physically means that the precessional motion of the magnetization is damped or, in other words, that the magnetic spins suffer a *relaxation* process. The value of the damping constant can be either determined by way of experiments, or evaluated from a statistical mechanics approach.

4.3.4. Interpretation of ${}^{\mathrm{D}}M^{\beta\alpha\gamma}$. It is difficult to figure out the effect resulting from the existence of the dissipative couple stress ${}^{\mathrm{D}}M^{\beta\alpha\gamma}$ upon the precession of the magnetization —as one can judge from the relative complexity of the (notwithstanding simple since linear) constitutive equation (4.29). Nevertheless, it can be said that, since the corresponding recoverable part ${}^{\mathrm{R}}M^{\beta\alpha\gamma}$ takes care of the classical exchange phenomena between neighbouring spins, ${}^{\mathrm{D}}M^{\beta\alpha\gamma}$ might take care of some associated internal friction phenomena, if any. In fact, we do not know of any approach in which the latter are described or accounted for but in the three-dimensional continuum theory given earlier by the author (Maugin 1972d). Therefore we do not go in the details of the analysis of the effect of the hypothetical existence of ${}^{\mathrm{D}}M^{\beta\alpha\gamma}$. Simply, we indicate the effect of the leading term. The terms involving the material 'constants' ξ and ν in equation (4.29) are at least of the order of c^{-2}. The predominant term in the expression (2.10) of $\mathscr{A}_{\beta\alpha\gamma}$ is that involving $\Omega_{\beta\alpha;\gamma}$. Hence we can consider ${}^{\mathrm{D}}M^{\beta\alpha\gamma} \simeq \zeta P^{\alpha\mu}P^{\beta\nu}P^{\gamma\lambda}\Omega_{\mu\nu;\lambda}$. Then it is shown, on account of equation (4.32), that the principal part of ${}^{\mathrm{D}}M^{\beta\alpha\gamma}{}_{;\gamma}$ is given by

$$ {}^{\mathrm{D}}M^{\beta\alpha\gamma}{}_{;\gamma} \simeq 2\frac{\delta L^2}{|\tilde{\mathscr{M}}_{\mathrm{S}}|}\tilde{\mathscr{M}}^{[\beta}\overset{*}{\Box}{}^2\dot{\tilde{\mathscr{M}}}{}^{\alpha]} \tag{4.38}$$

in which we have introduced, for reason of dimensionality, a characteristic length L, for instance, a typical lattice constant, and set

$$\delta(\rho, \theta, \tilde{\mathscr{M}}_{\mathrm{S}}) \equiv \frac{\zeta(\rho, \theta, \tilde{\mathscr{M}}_{\mathrm{S}})}{|\tilde{\mathscr{M}}_{\mathrm{S}}|L^2}.$$

$\overset{*}{\Box}{}^2$ is the invariant differential operator introduced below (equation (4.44)). Then, adding the term (4.38) to equation (4.34), we should get instead of equation (4.36):

$$\tilde{\pi}^\alpha \simeq -\gamma\{\mathscr{B}^\alpha_{\mathrm{eff}} + |\tilde{\mathscr{M}}_{\mathrm{S}}|^{-1}(\alpha\dot{\tilde{\mathscr{M}}}{}^\alpha + \delta L^2\overset{*}{\Box}{}^2\dot{\tilde{\mathscr{M}}}{}^\alpha)\} \tag{4.39}$$

which shows the supplementary alteration brought to the precession velocity. The term added is a relativistic generalization of the term introduced in the classical theory in order to take account of a similar dissipative phenomenon (cf equation (4.60) in Maugin 1972d).

4.3.5. Heat conduction. From equations (2.4) and (4.27), it is possible to derive a relativistic equation of heat conduction. On account of the definition of Φ (equation (4.1)), of the decomposition (2.16) and equation (3.2, part one), equation (2.4) yields the following heat propagation equation valid for any constitutive equation for $\hat{q}_\beta$:

$$\rho\theta\dot{\eta} + \nabla_\beta\hat{q}^\beta - \frac{1}{\theta}\hat{q}^\beta\overset{*}{\nabla}_\beta\theta - \Phi + \rho h = 0 \tag{4.40}$$

in which we have defined the projected operator of covariant differentiation $\overset{*}{\nabla}_\beta$ by

$$\overset{*}{\nabla}_\beta \equiv P^{\cdot\alpha}_{\beta}\nabla_\alpha = \nabla_\beta + \frac{1}{c^2}u_\beta u^\alpha\nabla_\alpha. \tag{4.41}$$

The expression (4.40) can be written in a more common form if we assume a constitutive equation of the form (4.27) for $\hat{q}_\beta$ and we specify ψ^*, for η is derived from ψ^* according to equation (3.2, part one). With ψ^* given by (3.1) and assumed to be at least of class C^2 in its arguments, it is possible to compute $\dot{\eta}$. Lengthy computations similar to those made in part II lead to the equation

$$\rho C_h\dot{\theta} + \nabla_\beta\hat{q}^\beta + \rho h = \left(\Phi + \frac{1}{\theta}\hat{q}^\beta\overset{*}{\nabla}_\beta\theta\right) + \mathfrak{T}^{\beta\alpha}(e_{\alpha\beta} - \Omega_{\alpha\beta}) + \mathfrak{A}^{\gamma\beta\mu}\mathscr{A}_{\beta\gamma\mu} \tag{4.42}$$

in which we defined

$$C_h \equiv -\theta\frac{\partial^2\psi^*}{\partial\theta^2} \geqslant 0$$

$$\mathfrak{T}^{\beta\alpha} \equiv \rho\theta\left(2\frac{\partial^2\psi^*}{\partial\theta\partial C_{KL}}x^{(\alpha}_{;K}x^{\beta)}_{;L} + \frac{\partial^2\psi^*}{\partial\theta\partial M_L}\tilde{\mathscr{M}}^\alpha x^\beta_{;L} + \frac{\partial^2\psi^*}{\partial\theta\partial M_{LK}}\tilde{\mathscr{M}}^\alpha_{;\mu}x^\mu_{;K}x^\beta_{;L}\right)$$

$$\mathfrak{A}^{\gamma\beta\mu} \equiv \rho\theta\frac{\partial^2\psi^*}{\partial\theta\partial M_{LK}}\tilde{\mathscr{M}}^{[\gamma}x^{\beta]}_{;L}x^\mu_{;K}.$$

We now assume that the material is *isotropic*. The values of the corresponding expressions of $\mathfrak{T}^{\beta\alpha}$ and $\mathfrak{A}^{\gamma\beta\mu}$ can be evaluated with a function dependence of ψ^* based on the invariants listed in (3.21). We shall not make these computations. The dissipation potential considered is given by equation (4.27). We also suppose that χ is a mere constant if the changes in temperature and density are not too large. Then

$$\nabla_\beta \hat{q}^\beta = -\chi\left(\nabla^\beta \overset{*}{\nabla}_\beta \theta + \frac{1}{c^2}\nabla_\beta(\dot{u}^\beta \theta)\right). \tag{4.43}$$

But it is possible to show that

$$\nabla^\beta \overset{*}{\nabla}_\beta \theta = \overset{*}{\Box}{}^2 \theta + \mathrm{O}(c^{-2}), \qquad \overset{*}{\Box}{}^2 \equiv P^{\alpha\beta}\nabla_\alpha \nabla_\beta = \overset{*}{\nabla}{}^\beta \nabla_\beta, \tag{4.44}$$

where the term $\mathrm{O}(c^{-2})$ contains proper time derivatives of θ of the first order only. The operator $\overset{*}{\Box}{}^2$ is essentially spatial for $P^{\alpha\beta}$ is the metric of the three-dimensional hypersurface (cf part I) expressed in four-dimensional notation. Further the quantity

$$(\Phi + (1/\theta)\hat{q}^\beta \overset{*}{\nabla}_\beta \theta)$$

does not depend on the gradient of θ except through terms of the order of c^{-2}. Thus equation (4.42) can be written as

$$\rho C_h \dot{\theta} + \rho h = \chi \overset{*}{\Box}{}^2 \theta + \left(\Phi + \frac{1}{\theta}\hat{q}^\beta \overset{*}{\nabla}_\beta \theta\right) + \mathfrak{T}^{\beta\alpha}(e_{\alpha\beta} - \Omega_{\alpha\beta}) + \mathfrak{A}^{\gamma\beta\mu}\mathscr{A}_{\beta\gamma\mu} + \mathrm{O}(c^{-2}). \tag{4.45}$$

If we take the nonrelativistic limit, for $c \mapsto \infty$, of this equation for a rigid nonmagnetized solid in absence of volumic heat source, then defining the thermal diffusibility by $\beta = \chi/\rho C_h$, equation (4.42) reduces to the classical Fourier equation

$$\frac{\partial \theta}{\partial t} = \beta \Delta \theta. \tag{4.46}$$

Equations (4.27) and (4.45) have been established here according to a logical scheme. Furthermore they reduce to well accepted results in the nonrelativistic limit. Nevertheless they are not entirely satisfactory from the relativistic viewpoint. Indeed it is known that the classical equation (4.46) represents a propagation of heat at infinite velocity. Heat perturbations propagate instantaneously. The same comments apply to the relativistic equation (4.45) for $\overset{*}{\Box}{}^2$, being essentially spatial, this equation contains no proper time derivatives of the temperature of an order higher than one. Then this equation is parabolic in θ. The associated propagation velocity is thus infinite. This is in direct contradiction with the bound imposed upon the propagation velocity of all physical phenomena in relativity. The assumption represented by equation (4.27) is therefore too naive. This fact was already remarked by several authors (eg Kranys 1966, Mahjoub 1971). The answer to this apparent paradox calls for the consideration of *functional* (and not function in the usual sense) constitutive equations for the relativistic heat flux four vector, equations that yield a *relaxation* process for the heat flux. For such attempts, we refer the reader to recent works of the author (Maugin 1973b, c).

Appendix 1

We consider quasi-linear dissipative constitutive equations. That is, the form of Φ given by equation (4.6) suggests we take the following 'natural' set of constitutive equations for dissipative phenomena:

$$
\begin{aligned}
&{}^{\mathrm{D}}t^{(\beta\alpha)} = L^{\beta\alpha\mu\nu}\sigma_{\mu\nu}, \qquad {}^{\mathrm{D}}t^{[\beta\alpha]} = \mathscr{L}^{\beta\alpha\mu\nu}v_{\mu\nu}, \qquad j^{\gamma} = \mathscr{P}^{\gamma\lambda}\mathscr{E}_{\lambda}, \\
&\hat{q}^{\beta} = \chi^{\beta\alpha}\overset{*}{\theta}_{\alpha}, \qquad \overset{*}{M}^{\mu\gamma} = \mathscr{Q}^{\mu\gamma\lambda\sigma}\overset{*}{\mathscr{A}}_{\lambda\sigma},
\end{aligned} \tag{A.1}
$$

in which the tensorial coefficients $L^{\beta\alpha\mu\nu}$, $\mathscr{L}^{\beta\alpha\mu\nu}$, $\mathscr{P}^{\gamma\lambda}$, $\chi^{\beta\alpha}$, and $\mathscr{Q}^{\mu\nu\lambda\sigma}$ depend on $\rho, \theta, \tilde{\mathscr{M}}_{\mathrm{S}}^2$ and the motion. We remark that all affinities $\dot{\alpha}_{(\beta)}$ and all fluxes $\chi_{(\beta)}$ are PU. It follows that the tensorial coefficients introduced are also PU, hence their *isotropic* values are expressed by means of the 'metric' $P^{\alpha\beta}$, that is, on account of the symmetries,

$$
\begin{aligned}
&L^{\beta\alpha\mu\nu} \equiv \lambda P^{\beta\alpha}P^{\mu\nu} + \mu(P^{\beta\mu}P^{\alpha\nu} + P^{\beta\nu}P^{\alpha\mu}), \\
&\mathscr{L}^{\beta\alpha\mu\nu} \equiv \alpha P^{\beta\alpha}P^{\mu\nu} + \beta P^{\beta\mu}P^{\alpha\nu} + \phi P^{\beta\nu}P^{\alpha\mu}, \\
&\mathscr{P}^{\gamma\lambda} \equiv \sigma P^{\gamma\lambda}, \qquad \chi^{\beta\alpha} \equiv -\chi P^{\beta\alpha}, \\
&Q^{\mu\gamma\lambda\sigma} \equiv \nu P^{\mu\gamma}P^{\lambda\sigma} + \zeta P^{\mu\lambda}P^{\gamma\sigma} + \xi P^{\mu\sigma}P^{\gamma\lambda},
\end{aligned} \tag{A.2}
$$

in which λ, μ, α, β, ϕ, σ, χ, ν, ζ and ξ are scalars which now depend only on $(\rho, \theta, \mathscr{M}_{\mathrm{S}}^2)$. Noting that $v_{\mu\nu}$ is skewsymmetric and $\overset{*}{\mathscr{A}}_{\lambda\sigma}$ is a general second order tensor for which one introduces the symmetric and skewsymmetric parts $\overset{*}{\mathscr{A}}_{(\lambda\sigma)}$ and $\overset{*}{\mathscr{A}}_{[\lambda\sigma]}$ (cf equations (4.20)), one gets from equations (A.1)

$$
\begin{aligned}
&{}^{\mathrm{D}}t^{(\beta\alpha)} = \lambda\sigma^{\gamma}_{\cdot\gamma}P^{\beta\alpha} + 2\mu\sigma^{\beta\alpha}, \\
&{}^{\mathrm{D}}t^{[\beta\alpha]} = \bar{\xi}v^{\beta\alpha}, \qquad \hat{q}^{\beta} = -\chi P^{\beta\alpha}\overset{*}{\theta}_{\alpha}, \qquad j^{\gamma} = \sigma\mathscr{E}^{\gamma}, \\
&\overset{*}{M}^{\mu\gamma} = \nu P^{\mu\gamma}\overset{*}{\mathscr{A}}^{\sigma}_{\cdot\sigma} + a\overset{*}{\mathscr{A}}^{(\mu\gamma)} + b\overset{*}{\mathscr{A}}^{[\mu\gamma]},
\end{aligned} \tag{A.3}
$$

where

$$
\bar{\xi} \equiv \beta - \phi, \qquad a \equiv \zeta + \xi, \qquad b \equiv \zeta - \xi.
$$

Carrying equations (A.3) in equation (4.6), we obtain a function homogeneous of degree two in the affinities $\dot{\alpha}_{(\beta)}$. Hence, after the equation which follows (4.15), the dissipation potential

$$
\tilde{\Phi} = \tfrac{1}{2}\Phi = \sum_{(\beta)} \chi_{(\beta)}\dot{\alpha}_{(\beta)}
$$

whose explicit form is given by equation (4.18).

Appendix 2

In part II (Maugin 1973a) and a preceding work (Maugin and Eringen 1972c) we have obtained several forms for the dynamical equation which governs the magnetic spin (in the nondissipative case). These are equations (II-1.9), (II-5.8) and equation (3.6) of Maugin and Eringen (1972c). That is, respectively,

$$
\dot{\tilde{\mathscr{M}}}_{\alpha} = \frac{1}{\mathrm{i}c}\epsilon_{\alpha\beta\mu\nu}\tilde{\mathscr{M}}^{\beta}\pi^{\mu}u^{\nu} + \frac{1}{c^2}u_{\alpha}\dot{u}_{\beta}\tilde{\mathscr{M}}^{\beta} \tag{A.4}
$$

$$
\mathsf{P}\{\dot{S}^{\alpha\beta}\} = \mathsf{P}\{2\gamma\tilde{\mathscr{M}}^{[\alpha}\mathscr{B}^{\beta]}_{\mathrm{eff}}\} \tag{A.5}
$$

$$
\dot{S}^{\alpha\beta} = 2\gamma\overset{*}{F}{}^{[\alpha}_{\cdot\mu}S^{|\mu|\beta]} \tag{A.6}
$$

with

$$S^{\alpha\beta} \equiv \frac{1}{\mathrm{i}c}\epsilon^{\alpha\beta\gamma\delta}\tilde{\mathscr{M}}_{\gamma}u_{\delta}, \qquad \pi^{\mu} \equiv \frac{1}{2\mathrm{i}c}\epsilon^{\mu\alpha\beta\lambda}\Omega_{\alpha\beta}u_{\lambda}. \tag{A.7}$$

In fact, since $\overset{*}{F}{}^{\alpha}_{.\mu}$ and $S^{\mu\beta}$ are PU, it was shown in part II that equations (A.5) and (A.6) were equivalent. The right hand sides of these equations are only alternate axial four-vector and dual skewsymmetric tensor forms. As to equation (A.5), it was established with equation (A.4) as starting point. They are in fact two different forms of one equation. Indeed, consider the equation

$$\dot{S}^{\alpha\beta} = 2\gamma\tilde{\mathscr{M}}^{[\alpha}\mathscr{B}^{\beta]}_{\mathrm{eff}} \tag{A.8}$$

and carry the first of (A.7) into its left hand side. Performing the proper time differentiation, we obtain

$$\frac{1}{\mathrm{i}c}\epsilon^{\alpha\beta\gamma\delta}(\dot{\tilde{\mathscr{M}}}_{\gamma}u_{\delta}+\tilde{\mathscr{M}}_{\gamma}\dot{u}_{\delta}) = 2\gamma\tilde{\mathscr{M}}^{[\alpha}\mathscr{B}^{\beta]}_{\mathrm{eff}}. \tag{A.9}$$

Now contract this with the expression $(1/\mathrm{i}c)\epsilon_{\alpha\beta\mu\nu}u^{\nu}$ and use equation (2.13) and the known results (cf part I)

$$u_{\alpha}u^{\alpha} = -c^2, \qquad \dot{u}_{\alpha}u^{\alpha} = 0, \qquad \tilde{\mathscr{M}}_{\alpha}u^{\alpha} = 0.$$

The resulting equation is

$$2\left(\dot{\tilde{\mathscr{M}}}_{\mu}-\frac{1}{c^2}\tilde{\mathscr{M}}_{\nu}\dot{u}^{\nu}u_{\mu}\right) = \frac{2\gamma}{\mathrm{i}c}\epsilon_{\alpha\beta\mu\nu}\tilde{\mathscr{M}}^{\alpha}\mathscr{B}^{\beta}_{\mathrm{eff}}u^{\nu}$$

or,

$$\dot{\tilde{\mathscr{M}}}_{\mu} = \frac{-\gamma}{\mathrm{i}c}\epsilon_{\mu\alpha\beta\nu}\tilde{\mathscr{M}}^{\alpha}\mathscr{B}^{\beta}_{\mathrm{eff}}u^{\nu} + \frac{1}{c^2}u_{\mu}\dot{u}_{\nu}\tilde{\mathscr{M}}^{\nu}. \tag{A.10}$$

Identifying term by term with equation (A.4), we see that the only possibility is

$$\pi^{\beta} \equiv -\gamma\mathscr{B}^{\beta}_{\mathrm{eff}}. \tag{A.11}$$

That is, as in the classical three-dimensional theory of micromagnetism (cf Maugin and Eringen 1972a), the four-dimensional angular velocity of the magnetization is nothing but the effective magnetic induction up to the gyromagnetic ratio†. Similarly, since $\Omega_{\alpha\beta}$ and $\overset{*}{F}_{\alpha\beta}$ are the relativistic duals of π^{α} and $\mathscr{B}^{\beta}_{\mathrm{eff}}$ respectively, we have

$$\Omega_{\alpha\beta} = -\gamma\overset{*}{F}_{\alpha\beta}. \tag{A.12}$$

This was the result given earlier by Maugin and Eringen (1972c)†. The three equations considered are therefore equivalent. However, while the first one appears as a pure kinematical relation, the two other ones are more instructive—they are true *dynamical* equations—for they provide an explicit form of the rotation velocity as well as the coupling with the stress–energy–momentum, that is, other fields such as the deformation field. Note also that equation (2.3) is a fourth form equivalent to equations (A.4) through (A.6).

† Also up to a sign. In Maugin and Eringen (1972a), γ is defined as minus the present γ, hence no minus sign in the corresponding formula.

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